❖ About "Duct acoustics"

- Industrial use
- Efficiency
- Predictability
- Complexity

Ducts are able to efficiently transmit sound over large distances!

Wave dynamics

1. Navier-Stokes equations

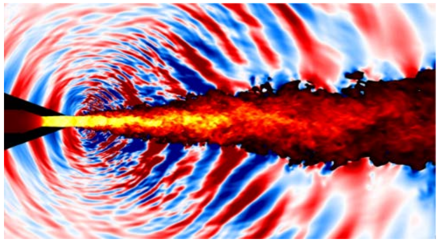
$$\frac{\partial \mathcal{Q}}{\partial t}+\frac{\partial \mathcal{F}_i}{\partial x_i}-\frac{\partial \mathcal{F}_i^v}{\partial x_i}=H \qquad \mathcal{Q}=\begin{bmatrix}\rho\\ u_i\\ e\end{bmatrix} \qquad \mathcal{F}_i=\begin{bmatrix}\rho u_i\\ \rho u_i u_j+p\delta_{ij}\\ (\rho e+p)u_i\end{bmatrix} \qquad \mathcal{F}_i^v=\begin{bmatrix}0\\ \tau_{ij}\\ u_k\tau_{ik}+q_i\end{bmatrix}$$

- Basic assumption for homogeneous linear formulation
  - Waves are free of non-linear effects and propagate with speed of sound
    ⇨ $\mathcal{Q}=\mathcal{Q}_0+\mathcal{Q}'$
  - Temperature change (heat transfer) is negligible
    ⇨ $q_i=0$
  - Viscous effects are sufficiently diminished in free space
    ⇨ $\tau_{ij}=0$
  - No external force (or source)
    ⇨ $H=0$

❖ Wave dynamics

2. Linearized Euler equations (2-D)

$$\frac{\partial \mathcal{Q}}{\partial t}+\frac{\partial \mathcal{F}}{\partial x}+\frac{\partial \mathcal{G}}{\partial x}=0 \qquad \mathcal{Q}=\begin{bmatrix}\rho' \\ u' \\ v' \\ p'\end{bmatrix} \quad \mathcal{F}=\begin{bmatrix}\rho_0 u'+\rho' u_0 \\ u_0 u'+p'/\rho_0 \\ u_0 v' \\ u_0 p'+\gamma p_0 u'\end{bmatrix} \quad \mathcal{G}=\begin{bmatrix}\rho_0 v'+\rho' v_0 \\ v_0 v' \\ v_0 u'+p'/\rho_0 \\ v_0 p'+\gamma p_0 v'\end{bmatrix}$$



- Fourier-Laplace transformation

$$\begin{bmatrix}\omega-k_x u_0-k_y v_0 & -k_x\rho_0 & -k_y\rho_0 & 0 \\ 0 & \omega-k_x u_0-k_y v_0 & 0 & -k_x/\rho_0 \\ 0 & 0 & \omega-k_x u_0-k_y v_0 & -k_y/\rho_0 \\ 0 & -\gamma k_x p_0 & -\gamma k_y p_0 & \omega-k_x u_0-k_y v_0\end{bmatrix}\begin{bmatrix}\tilde{\rho}' \\ \tilde{u}' \\ \tilde{v}' \\ \tilde{p}'\end{bmatrix}=0 \;\Rightarrow\; \begin{aligned}\lambda_1=\lambda_2&=\left(\omega-k_x u_0-k_y v_0\right) \\ \lambda_3&=\left(\omega-k_x u_0-k_y v_0\right)+c\left(k_x^{\,2}+k_y^{\,2}\right)^{1/2} \\ \lambda_4&=\left(\omega-k_x u_0-k_y v_0\right)-c\left(k_x^{\,2}+k_y^{\,2}\right)^{1/2}\end{aligned}$$

- Dispersion relation (Relationship between frequency and wavenumber)

$\lambda_1=\lambda_2=\left(\omega-k_x u_0-k_y v_0\right)=0$: Entropy and vorticity waves

$\lambda_3\lambda_4=\left(\omega-k_x u_0-k_y v_0\right)^2-c^2\left(k_x^{\,2}+k_y^{\,2}\right)=0$: Acoustic waves

$\omega^2-c^2\bar{k}^2=0$: Acoustic waves w/o mean flow ⇒ $c=\dfrac{\omega}{k}$:phase velocity

❖ Wave dynamics

3. Wave Equations

- Mass conservation

$$\frac{\partial \rho'}{\partial t}+\rho_0\frac{\partial u'}{\partial x}+u_0\frac{\partial \rho'}{\partial x}=0$$

- Momentum conservation

$$\frac{\partial u'}{\partial t}+u_0\frac{\partial u'}{\partial x}+\frac{1}{\rho_0}\frac{\partial p'}{\partial x}=0$$

- Assuming there is no mean flow: $\frac{\partial \rho'}{\partial t}-\frac{\partial p'}{\partial x^2}=0$,in 3D: $\frac{\partial \rho'}{\partial t^2}-c^2\nabla^2 p'=0$
- Assuming pressure is a function of density alone,

$$p=p_0+(\rho-\rho_0)\frac{dp}{d\rho}+(\rho-\rho_0)^2\frac{1}{2}\frac{d^2p}{d\rho^2}+\cdots \Rightarrow p-p_0=(\rho-\rho_0)\frac{dp}{d\rho}\Leftrightarrow \boxed{p'=\rho' c^2}$$

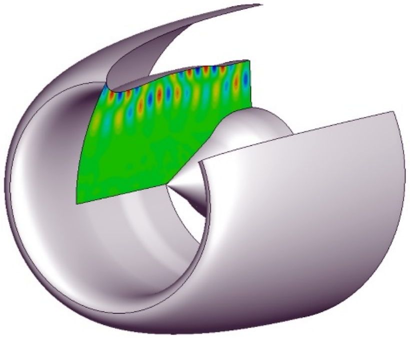
$$\Rightarrow \boxed{\frac{\partial p'}{\partial t^2}-c^2\nabla^2 p'=0}$$

: Only works when there is no mean flow!

❖ Duct acoustic problems

- Ventilation ducts
- Exhaust ducts
- Automotive silencers
- Shallow water channels and surface ducts in deep water
- Turbofan engine ducts

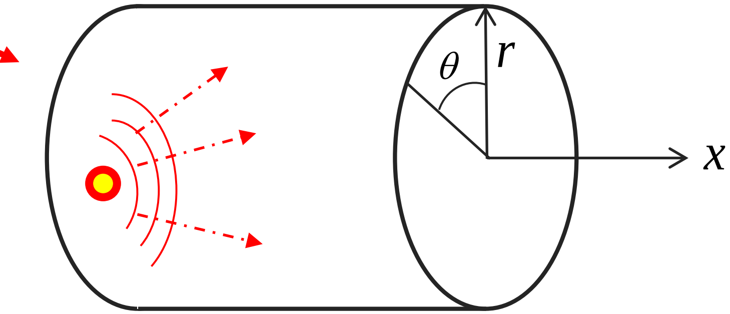
❖ How to Solve?



- Experiment
- Numerical simulation
- **Analytical approach**
  - Modelling to simple geometry (circular or rectangular ducts)
  - Appropriate assumptions (harmonic solution, no mean flow, linearization and so on..)
  - Dispersion relations
  - Mathematical functions
  - Boundary conditions

Cylindrical duct

- Modelling to simple geometry: simple cylindrical duct with hard wall



- Appropriate assumptions: assume harmonic separable solution

$$\frac{\partial p}{\partial t^2} - c^2 \nabla^2 p = 0$$, Cylindrical Laplacian: $$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$p(x, r, \theta, t) = X(x)R(r)\Theta(\theta)T(t).$$

- assuming harmonic separable solution,

$$\frac{X''}{X} + \frac{R'' + R'/r}{R} + \frac{\Theta''}{r^2\Theta} - \frac{1}{c^2}\frac{T''}{T} = 0.$$

- If we substitute derivatives over functions into some independent variables,

$$\frac{X''}{X} = -k^2, \quad \frac{\Theta''}{\Theta} = -m^2, \quad \frac{T''}{T} = -\omega^2 \quad \frac{R''}{R} = -n^2$$

$$\Rightarrow \quad r^2R'' + rR' + r^2(\omega^2/c^2 - k^2 - m^2/r^2)R = 0$$

Thus, general solution is implied as $p = R(r)e^{i(kx+m\theta-\omega t)}$

- From dispersion relation of acoustic waves, $\omega^2 - c^2\bar{k}^2 = 0$

$$\omega^2 - c^2\left(k^2 + \mu_{mn}{}^2\right) = 0 \quad , \quad \mu_{mn}{}^2 = k_m{}^2 + k_n{}^2$$

$$r^2R'' + rR' + (r^2\mu_{mn}{}^2 - m^2)R = 0$$

- Final equation resembles Bessel function

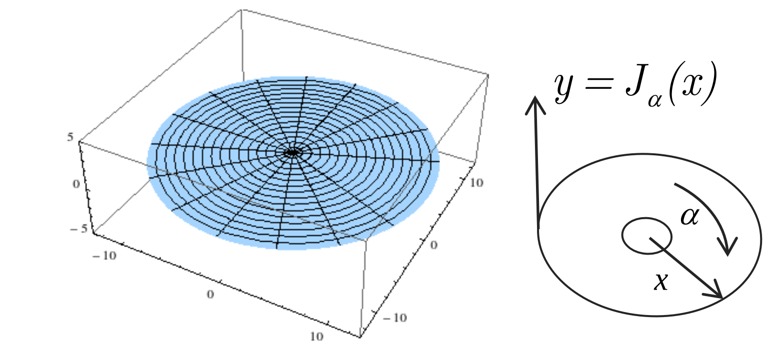
$$r^2R'' + rR' + (r^2\mu_{mn}{}^2 - m^2)R = 0$$

$$\tilde{r} = r\mu_{mn}$$

$$\tilde{r}^2R'' + \tilde{r}R' + (\tilde{r}^2 - m^2)R = 0$$

$$\boxed{R(r) = J_m(\tilde{r})}$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)\,y = 0$$



[Mode shapes of Bessel function]

- Neumann B.C. at the wall

$$\left.\frac{dR(r)}{dr}\right|_{r=a} \Rightarrow \boxed{J'_m(a\mu_{mn}) = 0}$$

- General solution of first derivative of Bessel function: α_{mn}

$$J'_m(\alpha_{mn}) = 0 \Rightarrow \boxed{\mu_{mn} = \frac{\alpha_{mn}}{a}}$$

- General Solution of Duct acoustics

$$p_{mn} = J_m\left(\frac{\alpha_{mn} r}{a}\right) e^{i(k_{mn}x + m\theta - \omega t)}$$

- From dispersion relation axial wavenumber become,

$$k = \sqrt{\frac{\omega^2}{c^2} - \mu_{mn}{}^2}$$

$$\Rightarrow \quad k_{mn} = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{\alpha_{mn}}{a}\right)^2} \qquad c_p = \frac{\omega}{k_{mn}} \quad \text{:Axial phase speed}$$

Note the condition for propagation of an acoustic mode is that the wave number k_{mn} must be real. Otherwise the wave will decay exponentially and is known as an evanescent wave. Therefore an $\{mn\}$ mode propagates if

$$\frac{\omega a}{c} > \alpha_{mn}. \quad \Rightarrow \quad \text{Cut-off frequency}$$

- Modes and mode shape functions

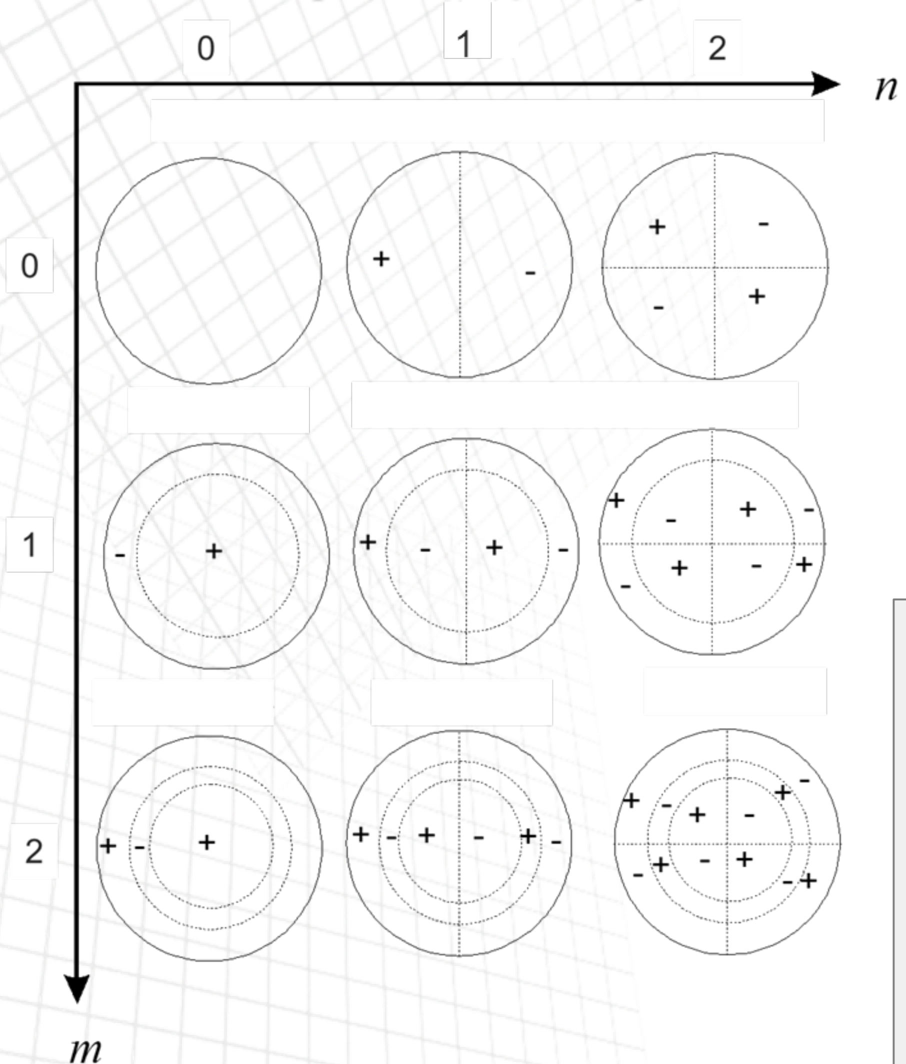
In seeking a solution for the pressure field in a duct we obtained, **not a single unique solution, but a family of solutions**. The general solution is a linear superposition of these 'eigenfunction' solutions:

$$p(r,\theta,x)=\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\bar{p}_{mn}\Phi_{mn}(r,\theta)e^{-i\alpha_{mn}kx}$$

$$\Phi_{mn}(k_{rmn}r)=J_m(k_{rmn}r)e^{im\theta}$$

The resultant acoustic pressure in the duct is the weighted sum of fixed pressure patterns across the duct cross section. Each of which propagate axially along the duct at their characteristic axial phase speeds.

- Mode shape functions of cylindrical duct



- Example

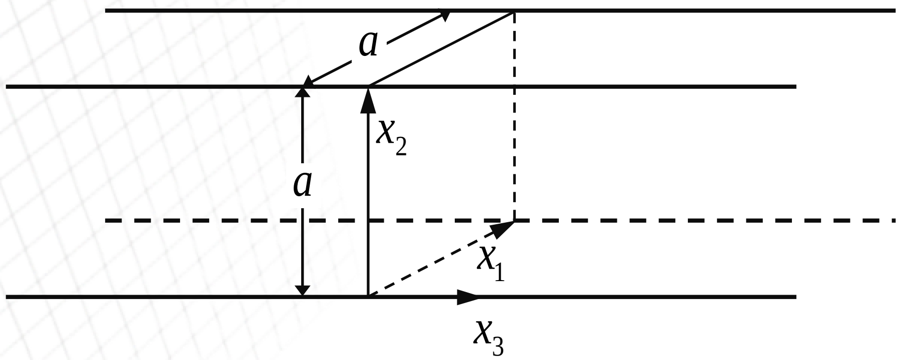
Consider a duct with radius a=0.5m, c=340m/s, sound frequency=3000rpm. Which modes will propagate?

The first roots of the derivative of the Bessel function

zero	$J_0'(x)$	$J_1'(x)$	$J_2'(x)$	$J_3'(x)$	$J_4'(x)$	$J_5'(x)$
1	3.8317	1.8412	3.0542	4.2012	5.3175	6.4156
2	7.0156	5.3314	6.7061	8.0152	9.2824	10.5199
3	10.1735	8.5363	9.9695	11.3459	12.6819	13.9872
4	13.3237	11.7060	13.1704	14.5858	15.9641	17.3128
5	16.4706	14.8636	16.3475	17.7887	19.1960	20.5755

$$\forall \alpha_{mn},\ J'_m(\alpha_{mn}) = 0$$

❖ Rectangular duct



- As an illustration, the sound of frequency ω in a rigid walled duct of square cross-section with sides of length a is considered

$$p'(\mathbf{x},t) = f(x_1)g(x_2)h(x_3)e^{i\omega t}$$

- With substitution for p' into the wave equation,

$$\frac{f''}{f} = -\frac{g''}{g} - \frac{h''}{h} - \frac{\omega^2}{c^2} = -\alpha^2$$

- Since a wall boundary condition is applied, function *f* is derived

$$f(x_1) = A_1 \cos\left(\frac{m\pi x_1}{a}\right) \quad , \quad \text{for some integer } m$$

- Similarly function *g* is derived

$$g(x_2) = A_2 \cos\left(\frac{n\pi x_1}{a}\right) \quad , \quad \text{for some integer } n$$

- Finally, function *h* is derived to the propagation form

$$h(x_3) = A_{mn} e^{-ik_{mn}x_3} + B_{mn} e^{ik_{mn}x_3} \qquad k_{mn} = \sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2}{a^2}\left(m^2 + n^2\right)}$$

- **The axial phase speed, $c_p = \omega / k_{mn}$ is now a function of the mode number and the propagation of a group of waves will cause them to disperse.**

- The pressure perturbation in the *(m,n)* mode has the form

$$p'(\mathbf{x},t)=\cos\left(\frac{m\pi x_1}{a}\right)\cos\left(\frac{n\pi x_2}{a}\right)\left[A_{mn}e^{-ik_{mn}x_3}+B_{mn}e^{ik_{mn}x_3}\right]e^{i\omega t}$$

- When k_{mn} is real, the pressure perturbation equation represents that waves are propagating down the x_3 axis with phase speed.

- When k_{mn} is purely imaginary, i.e. exceeds the cut-off frequency, the strength of mode varies exponentially with distance along the pipe. Such disturbances are evanescent

- Modes shape functions of rectangular duct

$$p'(\mathbf{x},t)=\cos\left(\frac{m\pi x_1}{a}\right)\cos\left(\frac{n\pi x_2}{a}\right)\left[A_{mn}e^{-ik_{mn}x_3}+B_{mn}e^{ik_{mn}x_3}\right]e^{i\omega t}$$

$$k_{mn}=\sqrt{\frac{\omega^2}{c^2}-\frac{\pi^2}{a^2}\left(m^2+n^2\right)}$$

$m=0, n=0$ $\quad$ $m=0, n=1$ $\quad$ $m=1, n=1$ $\quad$ $m=1, n=2$